PARAMETRIC FORM OF BOYER-LINDQUIST COORDINATES

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Abstract: Boyer-Lindquist coordinates are the most commonly used in the study of the Kerr black hole solution in the general theory of relativity. These coordinates are particularly used when describing the properties of a rotating (Kerr) black hole. In this study, I derived the parametric form of Boyer-Lindquist coordinates. In this article we described the line element in terms of parametric Boyer-Lidquist coordinates. I think it will be helpful in making calculations easier when solving black hole problems and drawing the related graphs.

Key words: General theory of relativity; Kerr black hole; Boyer Lindquist coordinates; parametric form.

1. Introduction:

Boyer Lindquist coordinates are a set of coordinates [1] –[19] generally used in the study of Kerr black hole solution in the theory of general relativity.They were introduced by the physicists H.Boyer and Richard W.Lindquist in the 1960s. The coordinates are particularly useful when describing the properties of a rotating (Kerr) black hole. In general relativity, the space-time around a massive object, like a black hole is described by a metric that defines the geometry of space-time. The Kerr metric, which describes a rotating black hole, is particularly complex and is most conveniently expressed using the Boyer-Lindquist coordinates. Boyer-Lindquist coordinates are adapted to the rotational symmetry of a Kerr black hole. They consist of three spatial coordinates (r, θ, and φ) and one time coordinate (t). These coordinates have the advantage of simplifying the expression of the Kerr metric and making it easier to study the behavior of objects and light near the black hole. These coordinates are crucial in understanding various phenomena around rotating black holes, such as the frame-dragging effect (Lense-Thirring precession), the ergosphere, and the properties of the event horizon. They play a vital role in the theoretical study of black hole physics and are an essential tool for making predictions and interpretations in the context of general relativity. In summary, Boyer-Lindquist coordinates are a set of spacetime coordinates specifically designed to describe the geometry and physics of a rotating black hole in general relativity, making them a fundamental concept in the study of these intriguing astronomical objects. Boyer-Lindquist coordinates provide a way to describe the geometry of a rotating black hole and the surrounding spacetime in a convenient and mathematically tractable manner. These coordinates take into account both the mass and angular momentum (rotation) of the black hole, making them well-suited for studying various physical phenomena in the vicinity of rotating black holes, such as the existence of an event horizon, ergosphere, and the properties of orbits of particles and light.

Boyer-Lindquist coordinates play a crucial role in the study of astrophysical black holes and have been instrumental in our understanding of the fascinating and complex physics associated with these cosmic objects.

2. Parametric form of Boyer Lindquist coordinates:

Line element:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - asin^2\theta d\emptyset)^2 + \frac{sin^2\theta}{\rho^2}\left((r^2 + a^2)d\emptyset - adt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

__________ (2.1)

Δ =*Discriminant* = $r^2 - 2Mr + a^2 + Q^2$

$\rho^2 = r^2 + a^2cos^2\theta$

$a = Kerr\ parameter = \frac{J}{M}$

Where Q = electric charge

M = mass of the black hole

J = angular momentum

We know that

$$\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=1$$

Relation between Boyer Lindquist coordinates and Cartesian coordinates:

$$x=\sqrt{r^2+a^2}\sin\theta\sin\emptyset \qquad (2.2)$$

$$y=\sqrt{r^2+a^2}\sin\theta\cos\emptyset \qquad (2.3)$$

$$z=r\cos\theta \qquad (2.4)$$

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$$x=\frac{a}{2}e^{-t}\sec t \qquad (2.5)$$

$$y=\frac{a}{2}e^{t}\sec t \qquad (2.6)$$

$$z=\frac{a\tan t}{\sqrt{2}}\sqrt{1-2\sinh^2 t} \qquad (2.7)$$

Since

$$\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=\frac{a^2\left(e^{2t}+e^{-2t}\right)\sec^2 t}{4a^2\sec^2 t}+\frac{(1-\sinh^2 t)}{2}$$

$$\Rightarrow\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=\frac{1}{2}[\cosh 2t+1-2\sinh^2 t]$$

$$\Rightarrow\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=\frac{1}{2}[\cosh^2 t+\sinh^2 t+1-2\sinh^2 t]$$

$$\Rightarrow \frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=\frac{1}{2}[1+cosh^2t-sinh^2t]=1$$

$$z = rcos\theta$$

$$cos\theta = \frac{z}{r}$$

From the definition of parametric form

$$cos\,\theta = \frac{1}{\sqrt{2}}\sqrt{1-2sinh^2t}$$

$$\Rightarrow sin\,\theta = \frac{1}{\sqrt{2}}\sqrt{1+2sinh^2t}$$

$$\Rightarrow sin\,\theta\;d\theta = \frac{-4\sin h\;t\cos h\;t}{\sqrt{2}\,\sqrt{1-2sinh^2t}}dt$$

$$\Rightarrow sin\,\theta\;d\theta = \frac{-2\sin h\;2t}{\sqrt{2}\,\sqrt{1-2sinh^2t}}dt$$

$$\Rightarrow sin^2\theta\;d\theta^2 = \frac{4\sin h\;t}{2\,(1-2sinh^2\;t)}dt^2$$

$$\Rightarrow d\theta^2 = \frac{4\sin h\;t}{2\,(1-2sinh^2\;t)sin^2\theta}dt^2$$

$$\Rightarrow d\theta^2 = \frac{2\sin h\ 2t}{(1-2sinh^2\ t)\cos h\ 2t}dt^2$$

$$\Rightarrow d\theta^2 = \frac{2\sin h\ 2t}{(1-4sinh^4\ t)}dt^2$$

$$\Rightarrow \rho^2 d\theta^2 = \frac{2\rho^2\ \sin h\ 2t}{(1-4sinh^4\ t)}dt^2 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (2.8)$$

(2.1)/(2.2)

$$\tan\emptyset = \frac{x}{y}$$

From parametric equations

$$\emptyset = \tan^{-1}(e^{-2t})$$

$$\Rightarrow d\emptyset = \frac{-2e^{-2t}}{1+e^{-4t}}dt$$

$$\Rightarrow d\emptyset = -\sec h(2t)dt \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (2.9)$$

$$\Rightarrow -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\emptyset)^2 = -\frac{\Delta}{\rho^2}\left(dt + \ a\frac{\cos h\ 2t}{2}\sec h\ 2t\ dt\right)^2$$

$$\Rightarrow -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\emptyset)^2 = -\frac{\Delta}{\rho^2}\left(1+\ \frac{a}{2}\right)^2\ dt^2 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (2.10)$$

$$\frac{\sin^2\theta}{\rho^2}\big((r^2+a^2)d\emptyset-adt\big)^2=\frac{\sin^2\theta}{\rho^2}(a^2\sec^2 t\,(-\sec h\,2t)dt-adt)^2$$

$$=\frac{a^2\cosh 2t}{2\rho^2}(a^2\sec^2 t\,(-\sec h\,2t)dt-adt)^2$$

$$\frac{\sin^2\theta}{\rho^2}\big((r^2+a^2)d\emptyset-adt\big)^2=\frac{a^2}{2\rho^2}(a\sec^2 t+\cosh\ 2t)^2dt^2 \quad\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (2.11)$$

$$\frac{\rho^2}{\Delta}dr^2=\frac{\rho^2}{\Delta}a^2sec^4t\ dt^2 \quad\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (2.12)$$

$$ds^2=C_1+C_2+C_3+C_4$$

$$C_1=-\frac{\Delta}{4\rho^2}(a+2)^2dt^2$$

$$C_2=\frac{a^2}{2\rho^2}(asec^2t+\cosh\ 2t)^2dt^2$$

$$C_3=-\frac{\rho^2}{\Delta}sec^4\ t\ dt^2$$

$$C_4=\frac{2\rho^2\ \sin h\ 2t}{(1-4sinh^4\ t)}dt^2$$

$$Where \quad \Delta = r^2 - 2Mr + a^2 + Q^2$$

$$\Rightarrow \Delta = a^2 sec^2\, t - 2aM\, tan\, t + Q^2$$

$$\rho^2 = \frac{a^2}{2}(2tan^2\, t - 2sinh^2\, t + 1)$$

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