MAGNIFICATION AND FRACTION GRACEFUL LABELING

AJENDRA KUMAR[1], AJAY KUMAR [2], NEERAJ GUPTA [3*]

ABSTRACT. For a (p,q) graph $G(V,E)$ with the labeling set of vertices and edges are A and B respectively. In this paper, the vertex set and edge set of G are labeled by lA and lB respectively, where $l \in \mathbb{N}$. We establish the relation between graphs labeled by sets A, B and lA, lB under certain conditions.

Keywords: Graceful labeling, l-Magnification graceful, odd graceful, prime graceful labeling, fraction graceful, α-fraction graceful.
Mathematics Subject Classification: 05C78

1. INTRODUCTION

The concept of graceful labeling was first given by Rosa [9] in 1967. Graceful labeling is an injective assignment of the positive integers of the vertices of a (p,q)-graph G from 0 to q and the edge is labeled bijectively by the modulus difference of the end vertices of the edges from 1 to q. Rosa [9] proved that path, caterpillar and cycle C_n can be labeled gracefully when $n \equiv 0, 3 \pmod 4$.
Acharya and Hegde [1] define f is injective assignment of vertex set to the $\{0, 1, 2, ..., k + (q-1)d\}$ and induced mapping $f^* : E(G) \longrightarrow \{k, k+d, ..., k+(q-1)d\}$ is bijective. Then f is called (k,d) graceful labeling, Where f^* is the modulus difference of end vertices of each edge. Gnanajothi [4] introduced odd graceful labeling in 1991, vertex and edge label defined in the set $\{0, 1, 2, 3, ...(2q-1)\}$ and $\{1, 3, 5, ...(2q-1)\}$ respectively. She proved that P_n, C_n, $K_{m,n}$, $P_n \odot K_1$ and crown $C_n \odot K_1$ for $n \equiv 0 \pmod 2$ are odd graceful and conjectured that tree graph are always odd graceful. Barrientos [2] proved that if all components of a forest are caterpillars, then it is odd graceful and a tree of maximum diameter five is odd graceful and conjectured that all bipartite graphs are odd graceful. Neela and Selvaraj [7] prove that the finite union of stars, paths and caterpillar, finite unions of ladders, the finite union of bi-star, paths and caterpillar, Corona graph $K_{m,n} \odot rK_1$ all are odd graceful. Gao [5] proved that the union of the finite number of paths, the union of the finite number of stars, the union of any number of paths and stars, $C_m \cup P_n$, $C_m \cup C_n$ and union of any number of cycles divided by 4 is odd graceful.
A. Tout et al. [11] introduced prime labeling. Prime labeling is defined as vertex set to $\{0, 1, 2, ..., p\}$ is injective and for each edge adjacent vertices should be co-prime. Path graph P_n, star graph $K_{1,n}$, cycle C_n, friendship graph F_n, $K_{m,2} \cup P_n$, bistar $B_{m,n}$, $C_4 \cup P_n$ and

*Corresponding Author - Neeraj Gupta
[1,3]Department of Mathematics and Statistics, Gurukul Kangri (Deemed to be University), Haridwar-249404 Uttarakhand, INDIA.
[2]Depaartment of Mathematics, Shaheed Srimati Hansa Dhanai Government Degree College, Agrora (Dharmandal), Tehri Garhwal, Uttarakhand.

$K_{m,2}$ are prime graceful [10]. Entringer [8] conjectured that all tree graphs are always prime graceful. Haxell et al. [6] proved that the large tree is prime graceful. Moreover, one can see the excellent survey of graph labeling by Gallian [3].

2. RESULTS

Graceful Labeling: Fraction and Magnification

Definition 2.1. *The function* $h : V(G) \longrightarrow \{0, \frac{1}{q}, \frac{2}{q}, ..., \frac{q}{q} = 1\}$ *is one-one mapping and induced function* $h^* : E(G) \longrightarrow \{\frac{1}{q}, \frac{2}{q}, ..., \frac{q}{q} = 1\}$ *is defined as* $h^*(xy) = |h(x) - h(y)|$ $\forall$ *edge* $xy = e \in E$ *is bijective mapping. Then* h *is called fraction graceful labeling on* G*.*

Definition 2.2. *If* h *is fraction graceful labeling on* G *and* $\exists\, r \in \mathbb{N}$ *such that* $h(x) \leq \frac{r}{q} < h(y)$ *or* $h(y) \leq \frac{r}{q} < h(x)$ $\forall$ *edge* $xy = e \in E$*, then* h *is called* α*-fraction graceful labeling on* G*.*

Definition 2.3. *The function* $h : V(G) \longrightarrow \{0, l, 2l, ..., ql\}$ *(where* $l \in \mathbb{N}$*) is one-one mapping and induced function* $h^* : E(G) \longrightarrow \{l, 2l, ..., ql\}$ *is defined as* $h^*(xy) = |h(x) - h(y)|$ $\forall$ *edge* $xy = e \in E$ *is bijective mapping. Then* h *is called* l*-magnification graceful labeling.*

Definition 2.4. *If* h *is* l*-magnification graceful labeling on* G *and* $\exists\, r \in \mathbb{N}$ *such that* $h(x) \leq lr < h(y)$ *or* $h(y) \leq lr < h(x)$ $\forall$ *edge* $xy = e \in E$*. Then* h *is called* l*-magnification-*α*-graceful labeling on* G*.*

Definition 2.5. *The function* $h : V(G) \longrightarrow \{0, l, 2l, ..., l(k + (q-1)d)\}$ *(where* $k, d \in \mathbb{N}$*) is one-one mapping and induced function* $h^* : E(G) \longrightarrow \{lk, l(k + d), ..., l(k + (q-1)d)\}$ *is defined as* $h^*(xy) = |h(x) - h(y)|$ $\forall$ *edge* $xy = e \in E$ *is bijective. Then* h *is called* l*-magnification-*(k, d)*-graceful labeling on* G*.*

Theorem 2.6. h *is graceful labeling on graph* G *if and only if* h *is* l*-magnification graceful labeling on graph* G*.*

Proof. Define $g : V(G) \longrightarrow \{0, l, 2l, ..., ql\}$ such that $g(x) = lh(x)$ $\forall\, x \in V$. If $g(x_1) = g(x_2)$ $\forall\, x_1, x_2 \in V \implies h(x_1) = h(x_2)$. Since h is graceful on $G \implies g$ is injective. Define induced mapping $g^* : E(G) \longrightarrow \{l, 2l, ...ql\}$ such that $g^*(xy) = |g(x) - g(y)|$ $\forall\, e = xy \in E$. $g^*(xy) = l|h(x) - h(y)| \implies g^*(xy) = lh^*(xy)$ (where h^* is induced mapping of h). Since h^* is bijective therefore g^* is bijective. Hence h is l-magnification graceful labeling.
Converse: putting $l = 1$. □

Corollary 2.7. h *is graceful labeling on* G *if and only if* h *is fraction graceful graceful on* G*.*

Proof. putting $l = 1/q$ in Theorem 2.6 □

Theorem 2.8. h *is* α*-graceful labeling on* G *if and only if* h *is* l*-magnification-*α*-graceful labeling on* G*.*

Proof. Define $g : V(G) \longrightarrow \{0, l, 2l, ..., ql\}$ such that $g(x) = lh(x)$ $\forall\, x \in V$. h is α-graceful labeling on G. So $\exists\, r \in \mathbb{N}$ and two disjoint set C and D such that $C \cup D = V \implies$ for any edge $xy = e \in E(G)$, $x \in C$, $y \in D$ such that $h(x) \leq r < h(y)$ or $h(y) \leq r < h(x)$. If we multiply by l then inequality becomes $g(x) \leq lr < g(y)$ or $g(y) \leq lr < g(x)$. Hence h is l-magnification α-graceful labeling on G.
Converse: Putting $l = 1$. □

Corollary 2.9. *h is α-graceful labeling on G if and only if h is α-fraction graceful labeling on G.*

Proof. Putting $l = 1/q$ in Theorem 2.8 □

Theorem 2.10. *h is (k,d)-graceful labeling on graaph G if and only if h is l-magnification-(k,d))-graceful labeling on graph G, where $k, d \in \mathbb{N}$, $l > 0$*

Proof. Define $g : V \longrightarrow \{0, l, 2l, 3l, ..., l(k + (q-1)d)\}$ such that $g(x) = lh(x)$. Since h is injective. So g is injective. Define induced mapping $g^* : E(G) \longrightarrow \{lk, l(k+d), l(k+2d), ..., l(k+(q-1)d)\}$ such that $\forall$ edge $e = xy \in E$, $g^*(xy) = |g(x) - g(y)|$. $g^*(xy) = lh^*(xy)$ where h^* is induced mapping of h. Since h^* is bijective. So g^* is bijective. Hence h is l-magnification-(k,d)-graceful labeling.
Converse putting $l = 1$. □

Theorem 2.11. *h is (k,d)-graceful labeling on G if and only if h is l-magnification-(k,d))-graceful labeling on G, where $k, d \in \mathbb{N}$.*

Proof. Define $g : V \longrightarrow \{0, l, 2l, ..., l(k+(q-1)d)\}$ such that $g(x) = lh(x)$. Since h is injective. So g is injective. Define induced mapping $g^* : E(G) \longrightarrow \{lk, l(k+d), l(k+2d), ..., l(k+(q-1)d)\}$ such that $\forall$ edge $xy = e \in E$, $g^*(xy) = |g(x) - g(y)|$. $g^*(xy) = lh^*(xy)$ where h^* is induced mapping of h. Since h^* is bijective. So g^* is bijective. Hence h is l-magnification-(k,d)-graceful labeling.
Converse putting $l = 1$. □

Odd Graceful Labeling: Fraction and Magnification

Definition 2.12. *The function $h : V(G) \longrightarrow \{o, \frac{1}{q}, \frac{2}{q}, ..., \frac{(2q-1)}{q}\}$ is one-one mapping and induced function $h^* : E(G) \longrightarrow \{\frac{1}{q}, \frac{3}{q}, ..., \frac{(2q-1)}{q}\}$ is defined as $f^*(xy) = |f(x) - f(y)|$ $\forall$ edge $xy = e \in E(G)$ is bijective. Then h is called odd fraction graceful labeling on G.*

Definition 2.13. *The function $h : V(G) \longrightarrow \{0, l, 2l, ..., (2q-1)l\}$ is one-one mapping and induced function $h^*E(G) \longrightarrow \{l, 3l, ..., (2q-1)l\}$ is defined as $h^*(xy) = |h(x) - h(y)|$ $\forall$ edge $xy = e \in E(G)$ is bijetive. Then h is called l-magnification odd graceful labeling on G.*

Theorem 2.14. *h is odd graceful labeling on graph G if and only if f is l-magnification odd-graceful labeling on G.*

Proof. Define $g : V(G) \longrightarrow \{0, l, 2l, ..., (2q-1)l\}$ such that $g(x) = lh(x)$ $\forall$ $x \in V(G)$. $g(x_1) = g(x_2) \implies h(x_1) = h(x_2)$. Since h is injective. So g is injective. Define induced mapping $g^* : E(G) \longrightarrow \{l, 3l, ..., (2q-1)l\}$ such that $g^*(xy) = |g(x) - g(y)|$ $\forall$ edge $xy = e$ $\in E(G) \implies g^*(xy = e) = lh^*(xy = e)$. g^* is bijective because h^* is bijective. Hence h is l-magnification odd graceful.
Converse: Putting $l = 1$. □

Corollary 2.15. *h is odd-graceful labeling on G if and only if h is odd-fraction graceful labeling on G.*

Proof. Putting: $l = 1/q$ in Theorem 2.14. □

Prime labeling: Fraction and Magnification

Definition 2.16. *The function $h : V(G) \longrightarrow \{l, 2l, ..., pl\}$ is bijective and $\forall$ edge $xy = e \in E(G)$, $\gcd(h(x), h(y)) = l$. Then h is called l-magnification prime labeling on G.*

Definition 2.17. *The function* $h : V(G) \longrightarrow \{lk, l(k+d), ..., l(k+(p-1)d)\}$ *(where* $k, d \in \mathbb{N}$*) is injective and* $\forall$ *edge* $xy = e \in E(G)$ *such that* $\gcd(h(x), h(y)) = l$*. Then* h *is called* l*-magnification-*(k, d)*-prime labeling on* G*.*

Theorem 2.18. h *is prime labeling on* G *if and only if* h *is* l*-magnification prime labeling on* G*.*

Proof. Define $g : V(G) \longrightarrow \{l, 2l, ..., pl\}$ such that $g(x) = lh(x)\ \forall\ x \in V(G)$. $g(x_1) = g(x_2) \implies h(x_1) = h(x_2)$. Since h is injective. So g is injective. For $\forall$ edge $xy = e \in E$, $\gcd(g(x), g(y)) = \gcd(lh(x), lh(y)) = l \gcd(h(x), h(y))$. Since h is prime labeling. So $\gcd(h(x), h(y)) = 1$. Hence h is l-magnification prime labeling.
Converse: Putting $l = 1$. □

Theorem 2.19. h *is* (k, d)*-prime labeling on* G *if and if* h *is* l*-magnification-*(k, d)*-prime labeling on* G*, where* $k, d \in \mathbb{N}$*.*

Proof. Prove as similar to the Theorem 2.18

□

References

[1] B. D. Acharya and S. M. Hegde, Arithmetics graphs, J. Graph Theory, 14 (1990) 275–299.
[2] C. Barrientos, Odd-graceful labelings of trees of diameter 5, AKCE International Journal of Graphs and Combinatorics, 6 (2009) 307-313.
[3] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics,(2020).
[4] R. B. Gnanajothi, Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, Madurai, (1991).
[5] Z. B. Gao, Odd graceful labelings of some union graphs, J. Nat. Sci. Heilongjiang Univ., 24 (2007) 35-39.
[6] P. Haxell, O. Pikhurko, and A. Taraz, Primality of trees J. Combinatorics, 2 (2011) 481-500.
[7] N. Neela and C. Selvaraj, Conjecture on odd graceful graphs, J. Combin. Math. Comput., 97 (2016) 65-82.
[8] O. Pikhurko, Trees are almost prime, Discrete Math., 307 (2007) 1455-1462.
[9] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internet Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, (1967), 349-355.
[10] T. M. Selvarajan and R. Subramoniam, International Journal of Engineering and Technology, 7 (2018) 750-752.
[11] A. Tout, A. N. Dabboucy, and K. Howalla, Prime labeling of graphs, Nat. Acad. Sci. Letters, 11 (1982) 365-368.